(Review of Previous Lecture)

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k}$$

is the frequency response of a system.

- amount of attenuation/gain for an input consisting of a complex exponential $e^{j\omega n}$ (sinusoid)
- can think of as a response for each frequency

Example: $H(e^{j\omega})$ for moving average system.

The impulse response of the moving average system is

$$h[n] = \frac{1}{M_1 + M_2 + 1}(u[n+M_1] - u[n-M_2-1]).$$

The frequency response is

$$\begin{aligned} H(e^{j\omega}) &= \frac{1}{M_1+M_2+1}\sum_{n=-M_1}^{M_2} e^{-j\omega n} \\ &= \frac{1}{M_1+M_2+1}\frac{e^{j\omega M_1} - e^{-j\omega(M_2+1)}}{1-e^{-j\omega}} \qquad \left(\because \sum_{k=N_1}^{N_2} a^k = \frac{a^{N_1} - a^{N_2+1}}{1-a}\right) \\ &= \frac{1}{M_1+M_2+1}\frac{\sin\omega\frac{M_1+M_2+1}{2}}{\sin\frac{\omega}{2}} e^{-j\omega\frac{M_2-M_1}{2}} \end{aligned}$$

Representations of Sequences by DTFT

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega, \quad \text{Inverse DTFT or synthesis equation}$$

Note: This is like a continuous version of $x[n] = \sum_{k=-\infty}^{\infty} a_k e^{-j\omega_k n}$

$$X(e^{j\omega}) \equiv \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}, \quad \text{DTFT or analysis equation}$$

Question: On a computer, do we compute $X(e^{j\omega})$ from a given $x[n]$ stored on a computer?

Answer: No !!

This requires infinite sequence $x[n]$ and produces continuous frequency variable ω. We can do this analytically for many signals as we will see in this course. But we cannot do it for real world signals. Instead on computer (e.g. MATLAB), we compute **DFT** (or **FFT**), which we will see later in this course.

Convergence and Wide-Sense DTFT

Recall that $X(e^{j\omega})$ is an infinite sum, hence may not converge. We need the convergence result, that is $|X(e^{j\omega})| < \infty$, for all ω.
Then,

$$|X(e^{j\omega})| = |\sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}| \leq \sum_{n=-\infty}^{\infty} |x[n]||e^{-j\omega n}| \leq \sum_{n=-\infty}^{\infty} |x[n]| < \infty$$

The last inequality is called absolute summability, which is a sufficient condition for the existence of $X(e^{j\omega})$.
Substitute $H(e^{j\omega})$ of some system for $X(e^{j\omega})$, we have the sufficient condition for stability of a LTI system: $\sum_{n=-\infty}^{\infty} |h[n]| < \infty$

Note: Any FIR with finite values. Hence, the system is stable.
Example:

$$x[n] = a^n u[n]$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} a^n u[n] e^{-j\omega n} = \sum_{n=0}^{\infty} (ae^{-j\omega})^n = \frac{1}{1 - ae^{-j\omega}}, \quad \text{if } |ae^{-j\omega}| = |a| < 1$$

The existence of FT requires

$$\sum_{n=0}^{\infty} |a|^n < \infty$$

which is the absolute summability as a sufficient condition.
Definition (Uniform Convergence) *Given some* $X(e^{j\omega})$ *and let*

$$X_M(e^{j\omega}) = \sum_{n=-M}^{M} x[n]e^{-j\omega n},$$

then FT of $x[n]$ *exists if* $\forall \varepsilon > 0$ *and* $\forall \omega$*, there exists* M_0 *such that*

$$|X(e^{j\omega}) - X_M(e^{j\omega})| < \varepsilon,$$

whenever $M > M_0$

That is, we have $\lim_{M \to \infty} |X(e^{j\omega}) - X_M(e^{j\omega})| \to 0$.

But, some sequences are not sbsolutely summable, so they don't achieve uniform convergence to some $X(e^{j\omega})$.

However, some sequences are square summable. For those that achieve **mean-square convergence**, we can give them FT's, too.

Definition (Mean Square Convergence) *Given some* $X(e^{j\omega})$ *and let*

$$X_M(e^{j\omega}) = \sum_{n=-M}^{M} x[n]e^{-j\omega n} ,$$

we say $X_M(e^{j\omega})$ *converges in mean-square sense if*

$$\lim_{M\to\infty} \int_{-\pi}^{\pi} | X(e^{j\omega}) - X_M(e^{j\omega}) |^2 \, d\omega \to 0 .$$

In this case, we say that $FT\{x[n]\} = X(e^{j\omega})$.

Note: we might have it that

$$| X(e^{j\omega}) - X_M(e^{j\omega}) | \text{ not goes to 0, as } M \to \infty$$

but the total energy of the difference between the two does, i.e., could be an infinite number of points that don't converge, but they have zero-measure.

The important example: Ideal low-pass filter

$$H(e^{j\omega}) = \begin{cases} 1, & |\omega| \le \omega_c \\ 0, & \omega_c < |\omega| \le \pi \end{cases}$$

The impulse response (by definition) is

$$h_{lp}[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} H(e^{j\omega}) e^{j\omega} d\omega = \frac{\sin \omega_c n}{\pi n}, \quad -\infty < n < \infty$$

The sequence $h_{lp}[n]$ is not absolutely summable. For checking the statement, let's see the following theorem.

Theorem (P series) $\sum_{n=1}^{\infty} \frac{1}{n^P}$ *converges if* $P > 1$*; diverges if* $P \le 1$.

By the above theorem, $h_{lp}[n]$ is not absolutely summable, but $h_{lp}[n]$ is square summable. That is,

$$\sum_{n=-\infty}^{\infty} \left| \frac{\sin \omega_c n}{\pi n} \right| \quad \text{diverges}$$

But,

$$\sum_{n=-\infty}^{\infty} \left(\frac{\sin \omega_c n}{\pi n} \right)^2 \text{ converges}$$

To see the FT is not uniform converges, we define

$$H_M(e^{j\omega}) = \sum_{n=-M}^{M} \frac{\sin \omega_c n}{\pi n} e^{-j\omega n} = FT\left\{ \frac{\sin \omega_c n}{\pi n} y[n] \right\} = \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} \frac{\sin[(2M+1)(\omega-\theta)/2]}{\sin[(\omega-\theta)/2]} d\theta$$

Different version of $H_M(e^{j\omega})$ are plotted in the textbook : **Figure 2.21** (p. 52.) The Gibbs phenomenon shows that $H_M(e^{j\omega})$ does not converge uniformly to $H(e^{j\omega})$ since the magnitude of the difference does not converge to 0 as $M \to \infty$.

But, we have shown that $h_{lp}[n]$ is square summable, this implies that

$$\lim_{M\to\infty}\int_{-\pi}^{\pi}\left|H(e^{j\omega}) - H_M(e^{j\omega})\right|^2 d\omega = 0$$

so, if places where values of two function are different have infinitely small spectral extent (zero-measure), the function $H(e^{j\omega})$ is OK as an FT.

Other important signals that are **neither absolutely summable nor square summable**. But, we have its FT.

Example Consider the sequence $x[n]=1,\ \forall n$. This sequence is neither absolutely summable nor square summable. Consequently, its FT does not converge in either the uniform or mean-square sense. However, it is possible and useful to define the Fourier transform of the sequence $x[n]=1,\ \forall n$ to be periodic impulse train

$$X(e^{j\omega}) = \sum_{r=-\infty}^{\infty} 2\pi\delta(\omega + 2\pi r)$$

Note I: the above impulse function $\delta(\cdot)$ is a function of a continuous variable and is of infinite height, zero width, and unit area, which consistent with the fact that $X(e^{j\omega})$ does not converge.

Note II: Since $X(e^{j\omega}) = \sum_{r=-\infty}^{\infty} 2\pi\delta(\omega + 2\pi r)$ is periodic. For finding its inverse FT, the integral extends only over one period. WLOG, we consider $r=0$. Then

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi} 2\pi\delta(\omega)e^{j\omega n}d\omega = 1,\ \forall n$$